Research Statement

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Background

An *environment* on $\mathbb{Z}^d$ is an assignment of transition probability vectors to each site $x \in \mathbb{Z}^d$. Formally, it may be defined as a function $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ with $\sum_{y\in\mathbb{Z}^d} \omega(x,y) = 1$ for all x. For a given environment ω and $x \in \mathbb{Z}^d$, we can define P_ω^x to be the law of a discrete-time Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}$, started at x, with transition probability from y to z given by $\omega(y,z)$. RWRE are random walks that occur in an environment that is itself randomly chosen. The environment-dependent measure P_ω^x is known as the *quenched law* of a random walk starting at x. For a probability measure P on the space of possible environments and a site $x \in \mathbb{Z}^d$, we may define an *annealed law* $\mathbb{P}^x$ by $\mathbb{P}^x(\cdot) = E_P\left[P_\omega^x(\cdot)\right]$. The probability of an event under $\mathbb{P}^x$ is the probability of the event if you know the walk starts at x but don't yet know what the environment looks like. A typical assumption is that under the measure P, transition probability vectors $(\omega(x, x+y))_{y\in\mathbb{Z}^d}$ at different sites x are independent and identically distributed (iid). In RWRE on $\mathbb{Z}$, the walk may be recurrent (meaning the walk returns to the origin infinitely many times with probability 1) or transient in one direction (meaning the limit of the walk is ∞ or $-\infty$). In the latter case, there is a $\mathbb{P}^0$–almost sure limiting speed $v \geq 0$. A surprising feature of RWRE is that $v = 0$ is possible even if directional transience holds. We say the walk is *ballistic* if $v > 0$.

Solomon [14] studied nearest-neighbor RWRE on $\mathbb{Z}$ in 1975, characterizing directional transience and calculating limiting speed in terms of simple, easily computable expectations involving the environment at a single site. One of his main results is the possibility that a walk is directionally transient but not ballistic. Kesten, Koslov, and Spitzer [7] then characterize Gaussian and non-Gaussian limit laws using a parameter κ, which they defined; they showed Gaussian limiting behavior for $\kappa \geq 2$ and non-Gaussian for $\kappa < 2$.

RWRE have proven quite challenging to analyze in settings other than the nearest-neighbor case of $\mathbb{Z}$. In the setting of RWRE on $\mathbb{Z}$ with bounded jumps (as opposed to nearest-neighbor jumps), directional transience and ballisticity have been characterized, but not in terms of parameters that can be directly computed. Nearest-neighbor RWRE on $\mathbb{Z}^d$ have also proven difficult to analyze, and although some sufficient conditions for directional transience and ballisticity have been studied, no general characterizations are known. In the setting of $\mathbb{Z}^d$, even a 0-1 law for directional transience under the iid assumption has yet to be proven. That is, if $\ell \in \mathbb{R}^d \setminus \{0\}$ and A_ℓ is the event that $\lim_{n\to\infty} X_n \cdot \ell = \infty$, it is still an open problem to show that $\mathbb{P}^0(A_\ell) \in \{0,1\}$ for $d \geq 3$. The case $d = 2$ was handled by Zerner and Merkl in [17].

Nevertheless, special cases of RWRE on $\mathbb{Z}^d$ have allowed for more to be proven. One such special case is random walks in Dirichlet environments (RWDE), where the transition probabilities vectors are assumed to be drawn from a Dirichlet distribution. Many conjectures that remain open for general nearest-neighbor RWRE on $\mathbb{Z}^d$ have been proven for RWDE. Not only has a 0-1 law been proven for all dimensions, but in the nearest-neighbor case, Sabot, Bouchet, and Tournier were able to characterize directional transience in terms of the Dirichlet weights assigned to the $2d$ different directions (see [11],[2],[15]). Moreover, if $d \geq 3$, all walks are known to be transient, even if they are not directionally transient. For $d \geq 3$, ballisticity has a simple characterization in terms of a parameter κ_0, which governs the strength of finite traps where the walk can get stuck (the walk is ballistic precisely when $\kappa_0 > 1$). This is a very different characterization from what is known for nearest-neighbor RWDE on $\mathbb{Z}$, where ballisticity is governed by the κ of [7]. This κ is always strictly smaller than κ_0 and reflects the existence of arbitrarily large intervals that are difficult to cross.

Past accomplishments

Ballisticity of RWDE on $\mathbb{Z}$ with bounded jumps [13]

Because Dirichlet environments had proven to be a fruitful model for $\mathbb{Z}^d$, I chose to study the question of ballisticity for RWDE on $\mathbb{Z}$ with bounded jumps. Let $L, R \geq 1$ be integers, and consider a vector of "Dirichlet weights" $(\alpha_i)_{i=-L}^{R}$, with each $\alpha_i \geq 0$. In my model, transition probabilities from site x to sites $x-L, \ldots, x+R$ are drawn according to a Dirichlet distribution with respective weights $\alpha_{-L}, \ldots, \alpha_R$, and this is done independently for each x.

I was able to define a parameter κ_1—a weighted sum of Dirichlet parameters—that is equivalent to the κ of [7] in dimension 1, but in the case of bounded jumps is no longer always greater than κ_0. It turns out that in order to achieve ballisticity, both κ_0 and $|\kappa_1|$ must be greater than 1. When $\kappa_0 \leq 1$, the walk has zero speed because of the relatively high likelihood of getting stuck in a region of bounded size for a long time. When $|\kappa_1| \leq 1$, the walk has zero speed because of the relatively high likelihood of repeatedly backtracking over regions of all sizes. These two types of slowing operate independently, and both parameters are necessary to look at, because the ordered pair $(\kappa_0, |\kappa_1|)$ can take on any value in the first quadrant of $\mathbb{R}^2$.

This is the first characterization of ballisticity for a class of RWRE on $\mathbb{Z}$ with bounded jumps in terms of parameters that can be directly computed.

As a step toward characterizing ballisticity for the Dirichlet model, I was able to give two abstract characterizations for general RWRE with bounded jumps on $\mathbb{Z}$. The first is that the walk is ballistic if and only if the expected time to reach the right of the origin is finite, and the second is that the walk is ballistic if and only if the expected number of total visits to the origin is finite. Similar characterizations have been given before, but under "ellipticity" assumptions not satisfied by the Dirichlet model (these assumptions have to do with certain transition probabilities or their ratios being almost surely bounded away from 0). After strengthening these abstract criteria to fit my model, I used tools that had been developed for the study of RWDE on $\mathbb{Z}^d$, together with coupling arguments and comparisons of several different modified measures on environments, to give my characterization in terms of κ_0 and κ_1. In fact, I was able to do more, fully characterizing the finiteness of moments under P (the measure on environments) for the expected time under the quenched measure P_ω^0 that the walk spends at the origin. Namely, if N_0 is the total number of visits to the origin, I showed that $E_P\left[E_\omega^0[N_0]^s\right] < \infty$ if and only if $s < \min(\kappa_0, |\kappa_1|)$. Setting $s = 1$ and applying my second abstract characterization then yields my main result.

Directional Transience of RWRE and RWDE on $\mathbb{Z}^2$ with bounded jumps [12]

The proofs of the 0-1 law for directional transience of RWRE on $\mathbb{Z}^2$ given in [17] and [16] rely on a nearest-neighbor assumption. I modified the proofs to remove that assumption, proving the 0-1 law for iid RWRE on $\mathbb{Z}^2$ with bounded jumps. This work was motivated by a desire to extend to the bounded-jump case the characterization of directional transience for nearest-neighbor RWDE on $\mathbb{Z}^d$ [10]. The proof of the 0-1 law for RWDE on $\mathbb{Z}^d$, $d \geq 3$, already worked for bounded jumps [2], [15], as did a proof that the walk is transient with positive probability in every direction ℓ with $\ell \cdot \mathbb{E}^0[X_1] > 0$ [15]. In addition to extending the 0-1 law for $d = 2$, I showed that for RWDE on $\mathbb{Z}^d$ where the annealed drift $\mathbb{E}^0[X_1]$ is zero, the projection of the walk onto any given direction is almost surely recurrent. Together, my two results were all that was needed to fully extend the characterization of directional transience for RWDE on $\mathbb{Z}^d$ from the nearest-neighbor case to the bounded-jump case.

Symbolic dynamics

I began my research in symbolic dynamics in the summer of 2015 at Hillsdale College under the supervision of Will Abram. I studied algebraic properties of n-ary interleaving operations on sets $X \subseteq \mathcal{A}^{\mathbb{N}}$ of symbol sequences with symbols drawn from a finite alphabet $\mathcal{A}$. One particular type of subset of $\mathcal{A}^{\mathbb{N}}$ is called a *path set*, and the class of path sets is closed under the interleaving operations I studied. The primary area of investigation for me was factorization: is it possible to say when a given path set can be obtained from interleaving some number of other path sets? In 2018, Jeffrey Lagarias, who had worked with Abram on previous papers related to path sets and interleaving, began working with me and with Abram to improve our results and exposition with the goal of publishing a paper. We ended up expanding the project and splitting our results into two papers, one of which ([1]) studies properties of interleaving operations on general subsets $X \subseteq \mathcal{A}^{\mathbb{N}}$, and one of which focuses on the special case of path sets. A main question for this work has to do with the structure of what we called *interleaving closure sets*: if, for a given $X \subseteq \mathcal{A}^{\mathbb{N}}$, $\mathcal{N}(X)$ is the set of n such that X can be factored as an n-fold interleaving of sets $X_0, \ldots, X_{n-1}$, which sets $N \subseteq \mathbb{N}^+$ have the property that there exists an $X \subseteq \mathcal{A}^{\mathbb{N}}$ with $\mathcal{N}(X) = N$? Another question has to do with iterated factorization. If a set X breaks down as the interleaving of some other sets X_i, and each of these breaks down as the interleaving of further sets $X_{i,j}$, and so on, is this iterated factorization process guaranteed to terminate? We were able to successfully answer these question, as well as others.

Future Work

Limiting distributions of RWDE with bounded jumps on $\mathbb{Z}$

One natural extension of the work I have done is to study limit theorems. Central limit theorems (CLTs) and stable limit laws have already been proven in [8], [6], and [5], but under uniform ellipticity assumptions not available in the Dirichlet case.

Proving central limit theorems (CLTs) requires characterizing, among other things, when certain quantities have finite second moments; in this case, these quantities will have to do with the amount of time it takes the walk to reach the right of the origin. Because one of my abstract characterizations of ballisticity is in terms of finite expected time to reach the right of the origin, and because the other involves moments that I have good control of, I hope to relate the two in a way that gives me a characterization of the finite second moments that I need; this will likely occur when $\min(\kappa_0, |\kappa_1|) > 2$. My hope is to then use methods similar to those of [8] and [6] to get CLTs, getting around their uniform ellipticity assumption by means of quantities that I am able to explicitly control in the Dirichlet case. After this, I can investigate the existence of limiting stable distributions in the cases where a CLT does not hold, potentially by modifying methods from [5].

Like my results on ballisticity, these results would be significant because the limiting distributions would be characterized in terms of quantities that one can often directly compute (κ_0 and κ_1), rather than limits involving norms of increasingly long products of matrices.

Computing κ_0

The parameter κ_0, which controls finite traps, is the minimal Dirichlet weight exiting a finite, strongly connected set of vertices. In the nearest-neighbor case of $\mathbb{Z}^d$, this minimum is always attained as the weight exiting a pair of adjacent vertices, and so κ_0 has a formula as a minimum of d different sums, each sum giving the Dirichlet weight exiting a pair of vertices adjacent in a particular dimension. In the bounded-jump case, the strongly connected sets with minimal exit weight are less restricted. I showed that given the set of $i \in \{-L, \dots, R\}$ for which $\alpha_i > 0$ (which is the set of i for which the walk may jump from x to $x + i$), there exists a formula for κ_0 as a minimum of finitely many sums of integer multiples of the α_i. However, I do not have a general way to find this formula, although I was able to find it in several specific instances. I was able to give an algorithm to compute κ_0 given L, R, and the actual values of the α_i, but my algorithm requires examining all subsets of a set of vertices that grows as the α_i change. If $L = R = 5$ and the weights α_i range from 0.1 to 2, my algorithm could require examining more than 2^{1500} sets of vertices. I would like to work on obtaining an efficient general algorithm to find the formula for κ_0 as an elementary function of the α_i, given only the set of i for which $\alpha_i > 0$. Although this study is motivated by probability, it involves more combinatorics and number theory than probability. Because I have been able to find this formula using *ad hoc* arguments in all the instances I've looked at, I am hopeful that I can find a way to turn the methods I've used into a general algorithm. A longer-term goal would be to find a similar algorithm for computing κ_0 for RWDE with bounded jumps in higher dimensions.

The significance of this work would be to enable easy computation of κ_0, and thus $\min(\kappa_0, |\kappa_1|)$ (since κ_1 is easy to compute)—a quantity which already is known to characterize ballisticity, and, as discussed above, may well be found to control limit theorems as well. This would mean the existence of a whole class of RWRE on $\mathbb{Z}$ (and potentially on $\mathbb{Z}^d$) with bounded jumps where one can easily and explicitly check ballisticity, and possibly limit laws.

Recurrence of balanced RWDE in $\mathbb{Z}^2$

For RWDE on $\mathbb{Z}^2$, if the annealed drift is nonzero, then the walk is transient in the direction of the expected first step. If the parameters are balanced, the walk is directionally transient with probability 0 in every direction.

In the latter case, the question remains open whether the walk is recurrent or transient;

that is, will it return to the origin infinitely many times or finitely many times? For $d = 1$, a walk with balanced parameters is recurrent [14], and for $d \geq 3$, such a walk is transient [9]. Because simple symmetric random walks on $\mathbb{Z}^d$ are recurrent for $d = 1, 2$ and transient for $d \geq 3$, it is conjectured that RWDE on $\mathbb{Z}^2$ with balanced parameters are in fact recurrent. I have spent some time attempting to prove this conjecture, and made progress using methods related to those in [9], where transience in the case $d \geq 3$ was proven. Currently, I have two steps missing. Roughly, I need to show that increasing Dirichlet weights along some path from one site x to another site y by a fixed positive amount increases the annealed probability that a walk started at x will reach y before returning to x. I can already prove it in the case where the increase is by an integer amount, but I need it to hold for arbitrarily small positive increases. The second thing I need to do is basically bound from below the probability that a walk started from a vertex connected to the edge of a box of radius N returns to this vertex $Nf(N)$ times before making its way to the center, for some increasing function f.

Ballisticity for general RWRE on $\mathbb{Z}$ with bounded jumps

In [13], I was able to show that for RWRE on $\mathbb{Z}$ with bounded jumps, the phenomena of finite trapping and large backtracking govern ballisticity. Either phenomenon on its own may be enough to cause zero speed (by having $\kappa_0 \leq 1$ or $\kappa_1 \leq 1$), but if neither is on its own enough to cause zero speed, then the walk is ballistic. An interesting question I posed in that paper is whether this is true for general RWRE on $\mathbb{Z}$ with bounded jumps. The phenomena of finite trapping and large backtracking can easily be defined in the general case, and it is interesting to ask whether it is true in general that the two kinds of slowing always operate independently, as they do for the Dirichlet case, or if it is possible for them to "conspire together" to cause zero speed, even if neither is strong enough to do so on its own. Prior to [13], conditions for ballisticity of RWRE on $\mathbb{Z}$ with bounded jumps (or RWRE on a strip, a generalization of the model) have required strong ellipticity assumptions that preclude finite trapping altogether (e.g., [3], [8]). Hence the relationship between the two forms of slowing when both are possible has not been studied. Therefore, an answer to this question would significantly deepen our understanding of ballisticity and of transient-but-not-ballistic behavior for RWRE on $\mathbb{Z}$.

Transience in a neighborhood vs. one direction only

Let $\ell \in S^{d-1}$ be a "direction" in the unit sphere in $\mathbb{R}^d$, $d \geq 2$. One can construct deterministic walks that are transient in direction ℓ (i.e., $X_n \cdot \ell$ approaches infinity) but recurrent in a direction ℓ' in every neighborhood of ℓ (i.e., $X_n \cdot \ell'$ comes arbitrarily near 0 infinitely often).

If A_ℓ^0 is the event that this behavior occurs, an interesting problem is to rule it out as an event of positive probability. Results on directional transience often rely on an assumption a positive probability of transience in all directions in a given neighborhood, rather than merely assuming positive probability of transience in a one direction (e.g., [4]). Proving that the probability of A_ℓ^0 is 0 for all directions ℓ would eliminate the need for this restriction. Moreover, as I pointed out in [12], proving the slightly stronger statement that the probability of an ℓ existing for which A_ℓ^0 occurs is 0 would allow for a more complete understanding of directional transience in, e.g, the Dirichlet case, where the probability of transience *in a given direction* has already been characterized, but when this probability is zero for every direction, the possibility of transience in a random direction has yet to be explored.

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