Teaching Statement

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From the time I roped my six-year-old sister into doing algebra with me, I have always been eager to share my love of mathematics with others. When I teach, my ultimate goal is for students to find satisfaction both in the beauty of the mathematical results they learn and in the human ingenuity required to discover and understand those results. I cannot directly make this happen for my students, but I always strive to do everything possible to create an atmosphere that is conducive to this goal. This does not happen automatically, but requires careful preparation and attention to detail, an understanding of how students think, and a great deal of empathy.

The reason for my goal

I believe God is not only the creator of the physical universe but also the author of mathematical truth. The author of Proverbs writes, in the voice of a personified Wisdom,

> "The LORD possessed me at the beginning of his work,
> The first of his acts of old.
> Ages ago, I was set up,
> At the first, before the creation of the world."
> (Proverbs 8:22-23, ESV)

Wisdom is therefore founded by God, along with all of creation. I believe wisdom includes mathematical truth; 1 Kings 4:33 talks about scientific knowledge as an aspect of Solomon's wisdom. Therefore, when we learn about mathematics, we are learning about God's work, and thus there is inherent goodness in it. In fact, because it is not restricted by the physical universe, mathematical truth is one object of study and one part of God's design that has not been marred by the Fall. I want students to find satisfaction in the beauty of mathematical results because mathematical truth is *good*.

Moreover, I believe using our reason to understand mathematical knowledge is part of what God created man to do. "It is the glory of God to conceal things, but the glory of kings is to search things out" (Proverbs 25:2). Our reason and understanding are part of what it means to be made in God's image, and using them to learn more about reality is part of the cultural mandate to subdue the earth, as God commanded in Genesis 1:28. This is a project far too large for one person alone; the command to subdue creation is given in the plural, to humanity as a whole. Thus, when we learn about the mathematics discovered or taught by the ancient Greeks, medieval Arabs, early modern Europeans, and so on, we are learning about and participating in a vast project of subduing the corner of God's dominion known as mathematics. I want students to find satisfaction in the ingenuity of others, and in using their own powers of reason, because this is part of how we fulfill our great calling as beings made in God's image.

Of course one does not need to be a Christian to do mathematics. God created all people in his image, and we all share in enjoying the goodness of creation, including the richness of mathematical inquiry, even if we do not fully recognize where this goodness comes from.

Emphasizing beauty, discovery, and understanding

I try to present material in a way that emphasizes beauty, discovery, and understanding, in addition to utility. When introducing a new formula or theorem, I focus on the reasoning behind it before delving into exercises that give practice using it (which are, to be sure, also very important).

When teaching the Fundamental Theorem of Calculus, for example, I believe it is important to let students appreciate the elegance of the fact that the derivative of $\int_a^x f(t)dt$ is simply $f(x)$ before showing them how this can be used to compute areas under curves. If a student becomes an expert at computing definite integrals before understanding the justification for the procedure, there is a very strong temptation not to care about the justification at all. If you already know how to solve this kind of problem, why care how the sausage is made? But if a student is first able to find satisfaction in understanding why the problem $\frac{d}{dx}\int_a^x f(t)dt$ has such a simple answer, the ability to use this fact to find exact values for areas under curves will only increase the student's appreciation for the beauty of the result.

Similarly, when explaining how to compute sums of geometric series, I always make sure to write out the first several terms of a series and let students see "infinitely many terms being canceled out," before introducing a formula like $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. That is, I will write an argument on the board in the following form.

$$\begin{array}{rcl} S & = & 1 + \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{9}} + \cancel{\tfrac{1}{27}} + \cancel{\ldots} \\ \frac{1}{3}S & = & \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{9}} + \cancel{\tfrac{1}{27}} + \cancel{\tfrac{1}{81}} \ldots \\ \hline \frac{2}{3}S & = & 1 \end{array}$$

This also provides an opportunity to make the same sly argument to "demonstrate" that $1 + 2 + 4 + 8 + \ldots = -1$, and then discuss the importance of thinking in terms of partial sums. The initial trick, which I've heard described as a "cancellation massacre," is clever and memorable, elegantly showcasing the main idea of the fully correct proof (and it is not mere slight of hand; each step can be justified under an a priori assumption of absolute convergence). The comparison with a spurious proof that $\sum_{n=0}^{\infty} 2^n = -1$ and the ensuing discussion can then help students better understand the nature of how mathematics deals with the infinite.

Giving the reasoning behind a result does not mean putting a rigorous proof on the chalkboard every time. Sometimes it may suffice to let students know that what they are seeing is just a higher-dimensional generalization of a theorem they already know. Often, a single picture that contains the crucial insight behind a theorem does more to help students see the elegance of a result and understand why it is true than a series of perfectly rigorous

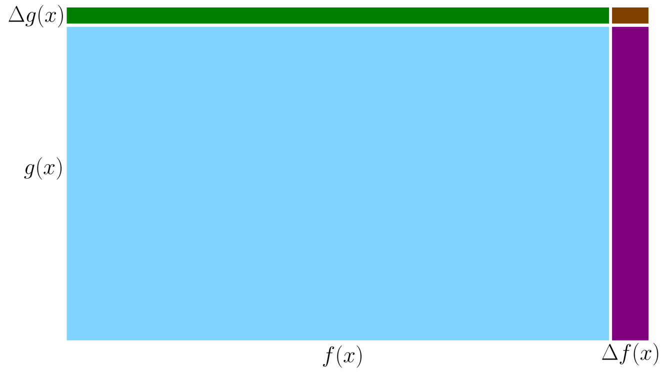

Figure 1: Product rule proof

calculations. This is true, for example, of the product rule. For an applied calculus class, a nicely labeled picture might convey enough of the reasoning that it is okay to move on, especially if time is short. If more time is available or if the class is more proof-based, showing the picture and then explaining how the picture leads to the calculation can be a good way to give students a peak into the difference between an insight and a rigorous proof, and how to transform one into the other.

Active involvement

I believe it is important for students to wrestle with mathematical questions themselves, and to participate as much as possible in the process of (re-)discovering what they are being taught. This process is inherently good and also contributes to understanding and retention.

Teaching in a flipped classroom

Some educators attempt to foster active learning by radically changing the structure of the class, such as with a "flipped classroom," where students watch lectures online and then spend their in-class time working together on problems. I have been a teaching assistant for two different flipped-classroom classes, and I have seen ways it can be more or less effective depending on how it is done.

In both flipped-classroom classes I have worked in, the goal is for students in groups of two to four to pool their brainpower to attack tough problems, helping each other find and understand answers. I find that explaining math to others solidifies my own understanding and enables me to take more ownership of the material, so when I see students helping each other and answering each other's questions, I know they are learning well. I have even seen multiple instances of students talking each other through frustrations, and it is always encouraging to see them caring for each other as people, rather than only focusing on their own work.

In one of the classes, we gave collaborative quizzes. It turns out these are one of the best ways to encourage collaboration, because there are tangible stakes involved, and students care for each other enough to want to help each other do well. This actually helps to separate the mathematics from the pursuit of a grade, since focusing away from oneself naturally makes one more receptive to enjoying the process of reasoning for its own sake.

Having seen the benefit of collaborative quizzes, I will consider using them in future classes. Of course, collaborative quizzes are less effective for assessment, including self-assessment, and remove one additional opportunity for students to practice solving problems on their own. The benefits therefore must be weighed against the disadvantages.

To foster a spirit of collaboration outside of quizzes, it is important to make sure each group is on the same page, and students do not move on ahead of their group-mates. To encourage this, as TAs we can make sure that if one member of a group has a question, another member of the group is able to explain what the question is to ensure that they have discussed it together. I have also seen that interrupting students frequently or showing them the solution to a problem after a short time discourages discussion, which thrives when students are given time and space. For a teacher, fostering productive collaboration therefore requires being comfortable with quiet and with relinquishing some control of the room.

Other helpful tactics I have seen include putting students together in the same group for several weeks at a time to help them get comfortable working together as a team, making attendance mandatory so that students are not left alone with their group-mates skipping class, and breaking up the hour with a short mini-lecture and cookies. I have also seen that students who voluntarily sign up for the flipped-classroom version seem to get more out of it than those who were not aware of the difference ahead of time.

Engaging students in a traditional classroom

Although a flipped-classroom format can be helpful, it is not the only way to actively engage students. I make it a priority to keep students as engaged as possible even in a traditional classroom. I often start the class with a warmup exercise. As I'm teaching, I frequently ask students to anticipate the next step. If I'm solving a problem they're expected to be able to solve (like a problem from a past quiz), it can be as simple as "Now what?"—followed up with "Good. Why?" For a new concept, some more prompting may be necessary.

Whenever I have the opportunity, I try to ask open-ended questions. These do not need to be hard; the goal is simply to provide an opportunity to think mathematically in a situation where there is not just one right answer that the teacher is looking for.

For a lesson on continuity, I like to use the following piecewise-defined function.

$$f(x) = \begin{cases} x^2+1 & \text{if} \quad x \leq 0 \\ \frac{\sin x}{x} & \text{if} \quad 0 \leq x \leq \frac{3\pi}{2} \\ e^x & \text{if} \quad \frac{3\pi}{2} < x \end{cases}$$

One can check that the function is continuous at 0 but not continuous at $\frac{3\pi}{2}$, because the limit from the left is $\frac{-2}{3\pi}$ and the limit from the right is $e^{\frac{3\pi}{2}}$. At this point, I like to ask students how we can be absolutely sure that those two limits really are different. Usually there's a pause as students realize that two very different-looking expressions could in fact be the same number, and it's not immediately obvious how to be sure they're different. Then after a few seconds, a bright student in the front will point out that e raised to any power is always positive, so if nothing else, one of the values is positive and the other is negative, meaning they're different. But this is not the only correct answer. A student might instead point out that e to a positive power is greater than 1, while $\frac{-2}{3\pi}$ has absolute value less than 1. A moment like this in the middle of class pulls students away from focusing on

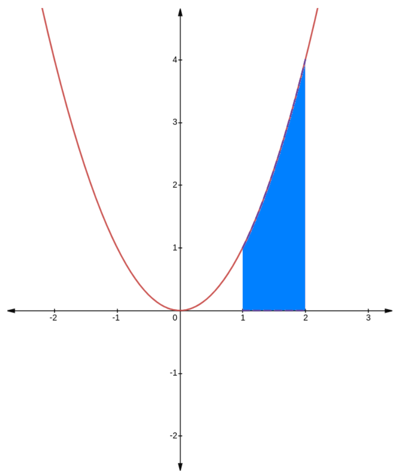

Figure 2: Estimating this area on the first day of a Calculus 1 class helps students appreciate the difficulty of finding an exact area while exercising their mathematical reasoning in an open-ended way.

learning definitions and solving problems that might appear on quizzes and tests and toward reasoning in a mathematical way for its own sake.

I like to do something similar at the beginning of a Calculus I semester, where I graph the parabola $y = x^2$ on the board, shade the area underneath it between $x = 1$ and $x = 2$, and ask students what they can tell me about the area. Usually, after a bit of thinking, someone points out that it must be at least 1, because it contains the square of height 1 between 1 and 2. Someone else soon chimes in that it must be less than 4, because it's contained in the 1×4 vertical box between 1 and 2. I challenge students to come up with better and better estimates.

Another technique I've tried to facilitate active learning is pairing students up to discuss a question before I tell them the answer. Even if they are not able to come up with the answer, the act of trying helps them to see why the problem is interesting and difficult, and helps them better appreciate the solution. I also try to have students present at the chalkboard when possible. Extra credit is a great incentive for this.

Choosing the right examples

"It is has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques. Indeed, it is only through examples that definitions have any meaning, since the technical words of mathematics describe classes of objects or relations with which the learner has to become familiar. . . . [L]earning mathematics can be seen as a process of generalizing from specific examples." [2]

The above quotation from an article by Anne Watson and John Mason does a good job summarizing what I believe about the importance of using examples in teaching. Because it is natural for human beings to move from specific examples toward more abstract knowledge, I make a point to begin with examples before introducing a novel concept. For instance, I will show students how to find the slope of a specific function, like $y = x^2$, at a specific point, like $x = 2$, before introducing the concept of the derivative $f'(x)$ for an arbitrary (differentiable) function $f(x)$.

Using examples is not enough, though; it is important to choose the right examples. I try to start with examples that are as simple as possible while making it clear why a definition exists. When I'm introducing the concept of a limit at a finite point, I don't start with $\lim_{x\to 0} \frac{\sin x}{x}$ because that's too complicated. But I also don't start with $\lim_{x\to 0} x^2$ or $\lim_{x\to 5} 3$. Even though these examples are simple, they fail to motivate the definition; why should we care about defining limits until we see an example where we can say something we couldn't say without it? Limits like $\lim_{x\to 5} 3$ can and should be used later, to reinforce the definition. But instead of beginning with them, I start with $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$. It is easy to see that plugging in 1 gives you $\frac{0}{0}$, and fairly easy to find the limit and explain why it is what it is. Also, the limit is not taken at 0, nor is the limit 0, so students are not tempted to restrict their thinking to cases involving 0.

Building students' confidence

If I feel like I don't understand something I'm supposed to understand, it's extremely hard to shake the feeling that I'm behind and playing catch-up whenever I see that thing come up again, and that feeling lingers even after I've understood it as well as anyone. When this happens to students, I think the best way to help is to acknowledge the feeling and empathize with the situation. A lot of the insecurity comes from feeling like things will be explained under the assumption that you have mastered something you haven't mastered. Knowing that you have a second chance to learn it from the beginning can restore confidence.

But of course, when possible, it is best to prevent the student from having that feeling in the first place. One way I try to prevent this is by always being honest; I think glossing over details for the sake of simplicity can lead students to feel like they are missing something. For example, when teaching separable differential equations, it is tempting to just tell students to "multiply both sides by dx and don't think about it." Of course this is wrong, because $\frac{dy}{dx}$ is not a real fraction, and dx is not a number that you can multiply both sides by. Even if we briefly explain that the shortcut is okay because it gives us a function that solves the equation, it is easy for a student to miss this and to feel out of the loop when trying to use the method. For this reason, I make sure to emphasize the seeming absurdity of the shortcut process and wait until everyone seems to be uncomfortable with it before showing it side-by-side with the technically correct process so that everyone can see why they yield the same result. Then they are able to use the shortcut on their homework with confidence.

It is important to me to make students feel like they're in the know as much as possible, even when it comes to conventions, notations, and names. I talk about how applying the "chain rule" repeatedly creates a *chain* of derivatives, and when integrals come up, I will show a picture of the original Bill of Rights that uses an elongated s so that they can see where the notation comes from.

Students can be intimidated by things as simple as different notations for the same thing,

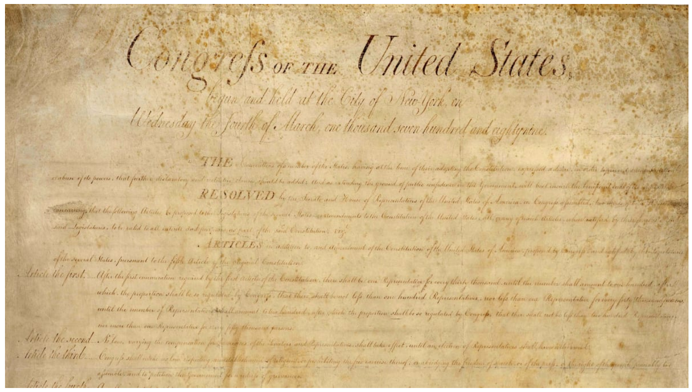

Figure 3: The original Bill of Rights contains a prominent elongated *s*.

or even Greek letters. I know because I used to be intimidated by Greek letters. To help prevent this, I acknowledge the potential for intimidation beforehand. When I'm teaching about Lagrange multipliers or eigenvalues, I'll ask people if they're scared of lambdas, say I used to be scared of Greek letters, and then tell them what I'm about to do normally uses a lambda but I'll use a k instead. Then I'll say "k or lambda" a few times as I go through the first problem. Usually by the time I'm finished with it, all the students have started to feel like I'm the one who's silly to be worried about Greek letters, and then I can switch to a lambda for the rest of the problems and no one minds.

Understanding specific students' needs

When teaching a class, it is important to understand that individual students might often have unique challenges and needs. I once had a student from another country ask me in the middle of a quiz, "How many feet are in a foot?" I realized that a lot of things I take for granted might not be obvious to people from different backgrounds. I try to be sensitive to such needs, for example by making sure that when I write quizzes or exam problems that require conversion between inches and feet, I include the fact that there are 12 inches in one foot, or by being aware that a simple typo in a word problem might be more likely to cause unnecessary confusion for someone who is not a native English speaker.

An important lesson I have learned is that students with unique needs or challenges can also bring unique contributions.

Technology

One of the best ways to teach math is to show helpful pictures. My favorite YouTube channel, 3Blue1Brown, is excellent at using pictures to elegantly convey mathematical ideas using beautiful animations. Although a skilled teacher can do a lot with chalk, the possibilities technology opens up to us are vast. I've often incorporated online 3D graphing or slope-plotting software while teaching multivariate calculus and differential equations, and I've used technology similarly in other classes as well.

However, I believe technology can also hurt students. For instance, graphing calculators can keep students from learning how to visualize functions on their own. Similarly, it is

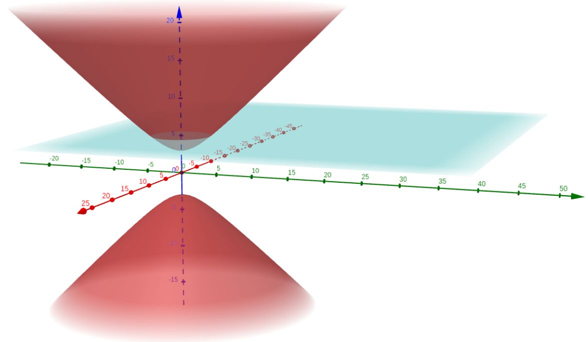

Figure 4: I found Geogebra to be an invaluable resource when teaching multivariate calculus.

easy to become over-reliant on scientific calculators. I have seen calculus students pull out a calculator to divide 150 by 3, or even to take the square root of 1. Reliance on calculators can prevent students from ever developing a healthy number sense. To be sure, there is value in calculators, as most real-world problems do not have clean solutions that always turn out to involve nice integers, fractions, and multiples of $\sqrt{2}$. Assigning problems that require calculators can help students to see how the math they are learning is used in the real world. But because of the danger of becoming dependent on calculators, I prefer to make these kinds of problems the exception rather than the rule, and encourage or require students to solve problems without calculators whenever possible.

Conclusion

I still have much to learn, and will continue to make myself a better teacher at my next institution. I look forward to learning more about the unique character of students there, studying how they tend to think and learn, and adjusting my teaching practices accordingly. I am eager to glean from what my colleagues have learned over their years of teaching. I also hope to read more math education literature and incorporate practices that are shown to be effective, and participate in Project NExT early in my career if I am able. Finally, I will continue to hone my practice as I gain experience. Just as there is always more math to learn, there is always more to learn about teaching, and my hope is that the day I stop growing as a teacher will be the day I retire.

References

[1] *The Holy Bible, English Standard Version*. Crossway, 2001.

[2] A. Watson and J. Mason. "Extending example spaces as a learning/teaching strategy in mathematics". In: *Proceedings of the 26rd Conference of the International Group for the Psychology of Mathematics Education*. (Norwich, UK). Ed. by Anne Cockburn and Elena Nardi. University of East Anglia, July 2002, pp. 378–385. ISBN: 0953998363.